JOURNAL OF APPROXIMATION THEORY 57, 223–238 (1989)

# $L_p$ Approximation from Nonconvex Subsets of Special Classes of Functions*

VASANT A. UBHAYA

*Department of Computer Science and Operations Research, North Dakota State University, Fargo, North Dakota 58105, U.S.A.*

*Communicated by Frank Deutsch*

Received June 9, 1987; revised October 16, 1987

An existence theorem for a best approximation to a function in $L_p$, $1 \leqslant p \leqslant \infty$, by functions from a nonconvex set is established under certain general conditions on the set. The unifying development and results are applicable to approximation from subsets of various classes of functions including quasi-convex, convex, super-additive, star-shaped, monotone, and $n$-convex functions. © 1989 Academic Press, Inc.

## 1. INTRODUCTION

Consider the problem of finding a best approximation from a nonconvex (i.e. not necessarily convex) set to a given function in the $L_p$ space of extended real functions defined on a compact real interval for $1 \leqslant p \leqslant \infty$. In this article, an existence theorem for a best approximation is established under certain general conditions on the subset. In addition, properties of $L_p$-bounded subsets are investigated. The results are applicable to $L_p$-approximation from subsets of various classes of functions including quasi-convex, convex, super-additive, star-shaped, monotone, and $n$-convex functions. Thus the analysis and results present a unifying development for special classes of $L_p$-approximation problems.

Let $I = [a, b]$, be a real interval and $H$ be the set of all extended real-valued functions on $I$. Let $L_p$, $1 \leqslant p < \infty$, denote the Banach space of all (equivalence classes of) Lebesgue measurable functions $f$ in $H$ with $\int |f|^p < \infty$ and norm $\|f\|_p = (\int |f|^p)^{1/p}$. Similarly, $L_\infty$ is the Banach space of (equivalence classes of) essentially bounded functions $f$ with norm $\|f\|_\infty = \text{ess sup } |f|$. Let $P \subset H$ be any (not necessarily convex) set. In what follows, a notation such as $P \cap L_p$ denotes all equivalence classes in $L_p$ to which a function in $P$ belongs. As usual, we shall carry out arguments for

* This research was supported by the National Science Foundation under Grant RII8610675.



0021-9045/89 $3.00
Copyright © 1989 by Academic Press, Inc.
All rights of reproduction in any form reserved.

a representative element of the equivalence class. Let $f \in L_p$, $1 \leqslant p \leqslant \infty$, and $\Delta$ denote the infinum of $\|f-k\|_p$ for $k$ in $P \cap L_p$. The problem under consideration is to find $g$ in $P \cap L_p$, called a best approximation to $f$ from $P \cap L_p$, so that

$$\Delta = \|f-g\|_p = \inf\{\|f-k\|_p : k \in P \cap L_p\}, \qquad 1 \leqslant p \leqslant \infty. \tag{1.1}$$

For $1 < p < \infty$, $L_p$ is uniformly convex and hence, a unique best approximation from $P \cap L_p$ exists if $P \cap L_p$ is closed and convex [1, 8]. However, we shall establish existence results for $1 \leqslant p \leqslant \infty$ under certain conditions on a nonconvex $P$.

We say that $P \subset H$ is sequentially closed if it is closed under pointwise convergence of sequences of functions. We denote by $\bar{P}$, the smallest superset of $P$ which is sequentially closed. Clearly, $P$ is sequentially closed if and only if $P = \bar{P}$. Immediately below, we state three conditions for a given $p$ with $1 \leqslant p \leqslant \infty$. Not all conditions will be imposed at the same time.

(1) $P \cap L_p = \bar{P} \cap L_p$. (This clearly holds if $P = \bar{P}$.)

(2) There exists a positive integer $z$ which depends upon $P$ only and the following holds: If $k \in P$, there exists an integer $1 \leqslant r \leqslant z$ and points $\{x_i : 0 \leqslant i \leqslant r\}$ with $a = x_0 < x_1 < \cdots < x_r = b$ so that $k$ is monotone (nondecreasing or nonincreasing) on each interval $(x_{i-1}, x_i)$. The integer $r$ and the points $\{x_i\}$, which are called the partitioning points of $k$, depend upon $k$.

(3) For every subset $B \subset P \cap L_p$ such that $\|k\|_p \leqslant D$ for all $k$ in $B$ for some $D > 0$, there exists a positive integer $r$ and points $\{x_i : 0 \leqslant i \leqslant r\}$ with $a = x_0 < x_1 < \cdots < x_r = b$, which depend upon $B$ only and the following holds: Functions in $B$ are uniformly bounded and have uniformly bounded total variation on every closed interval $[c, d] \subset \bigcup_i (x_{i-1}, x_i)$, the bounds possibly depending upon $[c, d]$.

Note that in condition (2) we do not assume that $k$ is alternatively nondecreasing and nonincreasing or vice versa on the intervals $(x_{i-1}, x_i)$. Although this is the structure displayed by examples given below, it is not necessary for analysis. Furthermore, the broad generality of the condition allows for one type of monotonicity (nondecreasing or nonincreasing) to exist in an interval without being restrained by types of monotonicity in other intervals.

We show in Section 2 that if $P$ satisfies conditions (1), (2) or (1), (3) for some $1 \leqslant p \leqslant \infty$, then $P \cap L_p$ is closed in $L_p$ and a best approximation to $f$ in $L_p$ from $P \cap L_p$ exists. Fundamental to this result is the following property of bounded sequences: If $(k_n)$ is an $L_p$-bounded sequence in $P \cap L_p$, then there exists a subsequence $(g_i)$ of $(k_n)$ and $g$ in $P \cap L_p$ such that $g_i \to g$ pointwise on $I$. In Section 3, we show that the stated conditions apply to the following classes of functions in $H$: Quasi-convex functions (denoted by $K_q$), convex functions ($K_c$), super-additive functions ($K_a$), star-shaped functions ($K_s$), monotone functions ($K_m$), and all $k$ which are $n$-convex on $(a, b)$ ($K_n$). Specifically, if $P \subset K_x$, where $x =$ q, c, s, m, and n, then $P$ satisfies condition (2) or condition (3) for all $p$. If, in addition, this $P$ satisfies condition (1) for some $p$, then the property of bounded sequences applies, $P \cap L_p$ is closed in $L_p$, and a best approximation from $P \cap L_p$ exists. In particular, these classes $K_x$ themselves satisfy condition (1) for all $p$ and, hence, the property of bounded sequences applies for all $p$, $K_x \cap L_p$ is closed in $L_p$, and a best approximation from $K_x \cap L_p$ exists. For subsets of $K_a$ these results hold under additional restrictions. We note that $K_q$ is a cone which is not convex and that $K_x$ for $x =$ c, a, s, m and n, are convex cones as subsets of $H$.

A general theory for existence of a best approximation in a normed linear space under certain given conditions is developed in [7]. According to the terminology of that article, the a.e. convergence in the $L_p$ spaces is the regular mode of sequential convergence. Furthermore, the property of bounded sequences given above is the property of boundedly a.e. sequential compactness of $P \cap L_p$. According to the results of [7], these conditions—the regular mode and boundedly compactness—are sufficient to ensure the existence of a best approximation. Thus, our conditions, which are designed to be applicable to the above special classes of functions, among others, in $L_p$, are stronger than those of [7] for a general normed linear space and imply, for these classes, the conditions of [7].

The methods of analysis are based on convergence properties of sequences of functions of bounded variation combined with special properties of classes of functions under consideration. These are extensions of methods used in the author's earlier work [23, 28] on quasi-convex and convex approximation. The isotone approximation problem in $L_p$, $1 < p < \infty$, which includes as a special case the monotone approximation problem, has been investigated in [12] by methods involving $\sigma$-lattices because of its special structure. However, these methods cannot be applied directly to our problem because of its more general setting involving several different classes of functions for which an underlying $\sigma$-lattice structure is not available. Existence of a best isotone and, hence, monotone $L_1$-approximation follows from Proposition 4 of [13]. Continuity of a best monotone $L_p$-approximation for $1 \leqslant p < \infty$ and its unicity for $p = 1$ under certain mild conditions on $f$ are established in [21] by a duality approach. Star-shaped and super-additive functions are analyzed in [3, 10, 19]. Uniform approximation by star-shaped, quasi-convex, convex and $n$-convex functions on a real interval or a subset of $R^n$ is considered in [24–27, 30], and least-squares approximation by quasi-convex functions in [29]. Basic references on $n$-convex functions or closely related classes of functions which are convex with respect to an $n$-parameter family or $\lambda(n)$-family of functions are [2, 4, 9, 11, 14, 22]. Uniform approximation by an $n$-parameter family is considered in [22]. Finally, certain approximation problems on $L_1$ are analyzed in [13, 20].

## 2. Existence of a Best Approximation

In this section we show that, under conditions (1), (2) or (1), (3) of Section 1, a best approximation from $P \cap L_p$ to a given $f$ in $L_p$ exists.

Lemma 2.1. *Assume condition* (2) *holds for* $P$. *Let* $k \in P \cap L_\infty$ *and let* $\{x_i: 0 \leqslant i \leqslant r\}$ *be the partitioning points of* $k$. *Then* $|k(s)| \leqslant \|k\|_\infty$ *for all* $s$ *in* $\bigcup \{(x_{i-1}, x_i): 1 \leqslant i \leqslant r\}$; $k$ *is possibly infinite on* $\{x_i\}$.

*Proof.* Assume $|k(s)| > \|k\|_\infty$ for some $s$ in $(x_{i-1}, x_i)$ for some $i$. Assume again that $k$ is nondecreasing on $(x_{i-1}, x_i)$; the proof for the case when $k$ is nonincreasing is similar. If $k(s) > \|k\|_\infty$, then $k(t) > \|k\|_\infty$ for all $t$ in $[s, x_i)$ and if $k(s) < -\|k\|_\infty$, then $k(t) < -\|k\|_\infty$ for all $t$ in $(x_{i-1}, s]$. This is a contradiction to the definition of $\|k\|_\infty$ and the proof is complete.

Lemma 2.2. *Assume condition* (2) *holds for* $P$ *for some positive integer* $z$. *Let* $(k_n)$ *be a sequence of functions in* $P \cap L_p$, $1 \leqslant p \leqslant \infty$, *such that* $\|k_n\|_p \leqslant D$ *for all* $n$ *and some* $D > 0$. *Then there exist an integer* $1 \leqslant r \leqslant z$, *points* $\{x_i, 0 \leqslant i \leqslant r\}$ *with* $a = x_0 < x_1 < \cdots < x_r = b$, *and a subsequence* $(h_j)$ *of* $(k_n)$ *with the following properties*:

(i) *If* $[u_i, v_i] \subset (x_{i-1}, x_i)$, $1 \leqslant i \leqslant r$, *then* $|h_j| \leqslant A$ *on* $\bigcup_i [u_i, v_i]$ *for all* $j \geqslant N$ *for some number* $A$ *and integer* $N$, *both of which depend upon the intervals* $[u_i, v_i]$.

(ii) *Each* $h_j$, $j \geqslant N$, *is monotone on* $[u_i, v_i]$.

*Proof.* We first consider the case $1 \leqslant p < \infty$. We show that $(h_j)$ satisfying (i) and (ii) exist. Let $\{x_{n,i}: 0 \leqslant i \leqslant r_n\}$ be the partitioning points of $(k_n)$. Since $1 \leqslant r_n \leqslant z$, some integer $r_n = r$ is repeated infinitely often, and, by compactness of $I$, some subsequence $x_{n_j,i}$, $0 \leqslant i \leqslant r$, of the partitioning points converges to $x_i$, $0 \leqslant i \leqslant r$, in $I$. Since $x_{n,i} < x_{n,i+1}$, we have $a = x_0 \leqslant x_1 \leqslant \cdots \leqslant x_r = b$. Some of the $x_i$ may be identical. We first assume that they are all distinct. Let $h_j = k_{n_j}$. Let $u_i' = (x_{i-1} + u_i)/2$ and $v_i' = (v_i + x_i)/2$. Choose $N$ so that $v_i' < x_{n_j,i} < u_{i+1}'$, $1 \leqslant i \leqslant r-1$, for all $j \geqslant N$. By condition (2), $h_j$ is monotone on each $[u_i', v_i']$ and (ii) follows. Consequently, $h_j(s) \leqslant h_j(u_i)$ for $u_i' \leqslant s \leqslant u_i$ and $h_j(s) \geqslant h_j(v_i)$ for $v_i \leqslant s \leqslant v_i'$ or reverse inequalities hold for $h_j$. Hence, if $\chi_i$ denotes the indicator function of $[u_i', u_i] \cup [v_i, v_i']$, it is easy to verify that

$$\begin{aligned} D &\geqslant \|h_j\|_p \geqslant \|h_j \chi_i\|_p \\ &\geqslant \max\{|h_j(u_i)|, |h_j(v_i)|\} \min\{(u_i - u_i')^{1/p}, (v_i' - v_i)^{1/p}\}. \end{aligned}$$

(To show this, if $h_j$ is nondecreasing on $[u_i', v_i']$, consider the following four cases: $h_j(u_i) \leqslant h_j(v_i) \leqslant 0$, $0 \leqslant h_j(u_i) \leqslant h_j(v_i)$, $0 \leqslant -h_j(u_i \leqslant h_j(v_i)$, and $h_j(u_i) \leqslant -h_j(v_i) \leqslant 0$.) It follows that, for $j \geqslant N$,

$$\max\{|h_j(u_i)|, |h_j(v_i)|\} \leqslant A_i, \qquad 1 \leqslant i \leqslant r, \tag{2.1}$$

for some $A_i$ which depends upon the intervals. Since $h_j$ is monotone on $[u_i, v_i]$, it is bounded there by the left side of (2.1); hence $|h_j| \leqslant A_i$ on $[u_i, v_i]$ for $j \geqslant N$. The number $A$ in (i) equals $\max\{A_i\}$. Now, if not all $x_i$ are distinct, then distinct $x_i$ may be reindexed and a similar argument as above may be applied.

If $p = \infty$, then let $\{x_i\}$, $(h_j)$, and $N$ be as in the case $1 \leqslant p < \infty$. Then (ii) holds. Again, by Lemma 2.1, $|h_j| \leqslant \|h_j\|_\infty \leqslant D$ on $\bigcup [u_i, v_i]$ for all $j \geqslant N$. Thus (i) holds with $A = D$. The proof is complete.

The following lemma may be proved by similar methods as above.

LEMMA 2.3. *Assume condition* (2) *holds for* $P$. *Let* $k \in P \cap L_p$, $1 \leqslant p < \infty$, *and let* $\{x_i : 0 \leqslant i \leqslant r\}$ *be the partitioning points of* $k$. *If* $[u_i, v_i] \subset (x_{i-1}, x_i)$, $1 \leqslant i \leqslant r$, *then* $|k|$ *is bounded on* $\bigcup_i [u_i, v_i]$. *Consequently,* $k$ *is finite on* $\bigcup_i (x_{i-1}, x_i)$.

THEOREM 2.1. *Assume conditions* (1), (2) *or* (1), (3) *hold for* $P$ *for some* $1 \leqslant p \leqslant \infty$. *Let* $(k_n)$ *be a sequence of functions in* $P \cap L_p$, $1 \leqslant p \leqslant \infty$, *such that* $\|k_n\|_p \leqslant D$ *for all* $n$ *and some* $D > 0$. *Then there exists a subsequence* $(g_j)$ *of* $(k_n)$ *and a* $g$ *in* $P \cap L_p$ *such that* $(g_j)$ *converges to* $g$ *pointwise on* $I$ *and* $\|g\|_p \leqslant D$.

*Proof.* Let conditions (1) and (2) hold for $P$ for some $p$. We assume first that $1 \leqslant p < \infty$. By Lemma 2.2, there exist points $x_i$, $0 \leqslant i \leqslant r$. and a subsequence $(h_j)$ of $(k_n)$ with the properties stated there. Let $0 < \varepsilon < \min\{x_i - x_{i-1} : 1 \leqslant i \leqslant r\}/2$ and for each positive integer $m$, let

$$I_m = \bigcup \{[x_{i-1} + \varepsilon/m, x_i - \varepsilon/m] : 1 \leqslant i \leqslant r\}.$$

We let $\chi_m$ be the indicator function of $I_m$. By Lemma 2.2, we infer existence of $A_m$, $N_m$ so that $|h_j| \leqslant A_m$ on $I_m$ for all $j \geqslant N_m$. Again, by the lemma, $h_j \chi_m$ is monotone on each interval $[x_{i-1} + \varepsilon/m, x_i - \varepsilon/m]$ and zero elsewhere.

Hence, the total variation of $h_j\chi_m$ does not exceed $4rA_m$, as may be easily verified. Thus, for each $m$, $(h_j\chi_m)$, $j = 1, 2, \ldots$, is a sequence of functions which is uniformly bounded and has uniformly bounded total variation. Hence, by Helly's selection theorem [15, p. 222] a subsequence $(f_{1i}\chi_1)$, $i = 1, 2, \ldots$, of the sequence $(h_j\chi_1)$ converges pointwise to a function $f_1$ on $I$ which is bounded by $A_1$. Again, by the same argument a subsequence $(f_{2i}\chi_2)$, $i = 1, 2, \ldots$, of $(f_{1i}\chi_2)$ converges pointwise to a function $f_2$ on $I$ which is bounded by $A_2$. We apply this diagonal procedure successively for each $m$. Since $I_m \subset I_{m+1}$ we have $f_m = f_{m+1}$ on $I_m$. We define a function $\psi$ on $\bigcup_m I_m = \bigcup_i (x_{i-1}, x_i) = J$ by $\psi(s) = f_m(s)$ if $s \in I_m$. Clearly $\psi$ is well defined and the diagonal sequence $(\psi_i = f_{ii})$ converges pointwise to $\psi$ on $J$. Again a subsequence $(g_j)$ of $(\psi_i)$ converges on $\{x_i\}$ possibly to $\pm\infty$. Thus $(g_j)$ converges pointwise on $I$ to an extended real function $g$ where $g = \psi$ on $J$ and $g$ is finite there.

We show that $g \in L_p$. Since $(g_j)$ is a subsequence of $(h_j)$, we have $\|g_j\chi_m\|_p \leqslant \|g_j\|_p \leqslant D$ and $|g_j\chi_m| \leqslant A_m$ for all sufficiently large $j$. Because of finiteness of measure, constant functions are in $L_p$. Hence letting $j \to \infty$ in $\|g_j\chi_m\|_p \leqslant D$ and using the dominated convergence theorem we have $\|g\chi_m\|_p \leqslant D$. Now $|g\chi_m|^p \uparrow |g|^p$ as $m \to \infty$ on $J$. Hence we conclude that $\|g\|_p \leqslant D$ by the monotone convergence theorem. Thus $g \in L_p$. Since $g_j \to g$ and $\bar{P}$ is sequentially closed we have $g \in \bar{P}$. Thus $g \in \bar{P} \cap L_p$ and it follows by condition (1) that $g \in P \cap L_p$. If $p = \infty$, then we may prove the result as above using Lemma 2.2 and Helly's selection theorem.

Now assume that conditions (1) and (3) hold for $P$ for some $p$. Then, by condition (3), functions in $B = \{k_n\}$ are uniformly bounded and have uniformly bounded total variation on $I_m$ defined earlier. The rest of the proof is similar to the one given above and is applied to $(k_n)$ instead of $(h_j)$. The proof is complete.

We remark that for the first case involving conditions (1) and (2) in the above proof, since $g$ is the limit of $(g_j)$, it consists of monotone segments as in condition (2). The set of partitioning points of $g$ is contained in $\{x_i\}$ and not every $x_i$ is necessarily a partitioning point of $g$. This is because adjacent partitioning points may coalesce in the limit.

THEOREM 2.2. *Assume conditions* (1), (2) *or* (1), (3) *hold for some* $p$, $1 \leqslant p \leqslant \infty$. *Then* $P \cap L_p$ *is closed in* $L_p$, *and a best approximation to* $f$ *in* $L_p$ *from* $P \cap L_p$ *exists.*

*Proof.* Let $1 \leqslant p < \infty$. We first show that $P \cap L_p$ is closed. Let $(k_n)$ be a sequence in $P \cap L_p$ such that $\|k_n - k\|_p \to 0$ for some $k$ in $L_p$. We show that $k \in P \cap L_p$. Indeed, there exists a subsequence $(h_n)$ of $(k_n)$ such that $h_n \to k$ a.e. Since $\|h_n\|_p$ are bounded, by Theorem 2.1, there exists a subsequence $(g_j)$ of $(h_n)$ such that $g_j \to g$ for some $g$ in $P \cap L_p$. Hence $k = g$ a.e. and $P \cap L_p$ is closed.

To show the existence of a best approximation, let $\Delta$ be as in (1.1) and let $(k_n)$ be a sequence in $P \cap L_p$ such that $\|f - k_n\|_p \to \Delta$. Then $(k_n)$ is $L_p$-bounded and, by Theorem 2.1, there exists a subsequence $(g_j)$ of $(k_n)$ and $g$ in $P \cap L_p$ such that $g_j \to g$ pointwise on $I$. By Fatou's lemma, we have $\|f - g\|_p \leqslant \liminf \|f - g_j\|_p = \Delta$. Thus $g$ is a best approximation. The proof for $p = \infty$ is simpler. The proof is complete.

We note that the metric projection and the set of all best approximations to a given $f$ for problem (1.1) have properties as stated in Theorem 2.7 of [7].

## 3. Special Classes of Functions

In this section we define special classes of functions in $H$ and show that conditions (1), (2) or (1), (3) of Section 1 apply to each of them.

### (i) *Quasi-Convex Functions*

A function $k$ in $H$ is quasi-convex if

$$k(\lambda s + (1 - \lambda)t) \leqslant \max\{k(s), k(t)\}, \tag{3.1}$$

holds for all $s$, $t$ in $I$, and all $0 \leqslant \lambda \leqslant 1$ [16, 17]. Let $K_q$ denote this class of functions. The proof of the following proposition is similar to that of Satz 5 of [6] or Proposition 2.1 of [26].

PROPOSITION 3.1. *$k$ is quasi-convex if and only if there exists an $x$ in $I$ such that $k$ is nonincreasing on $[a, x)$ ($[a, x]$) and nondecreasing on $[x, b]$ ($(x, b]$).*

Note that, in the above proposition, the partitioning point $x$ may equal $a$ or $b$. Clearly, condition (2) applies to any $P \subset K_q$ with $z = 2$ and $r = 1$ or 2. If $r = 1$ then $k$ is monotone, nondecreasing, or nonincreasing on $I$. Let $(k_n)$ be a sequence in $K_q$ such that $k_n \to k$ pointwise on $I$. Then by (3.1), $k \in K_q$. Hence, $K_q = \bar{K}_q$. Thus, conditions (1) and (2) hold for $P = K_q$ for all $1 \leqslant p \leqslant \infty$. If $k \in K_q \cap L_p$, any $1 \leqslant p \leqslant \infty$, then by Proposition 3.1, Lemma 2.1, Lemma 2.3, and a simple argument we have $k > -\infty$ on $[a, x) \cup (x, b]$, $k < \infty$ on $(a, b)$, $k(a) < \infty$ if $x = a$, and $k(b) < \infty$ if $x = b$.

### (ii) *Convex Functions*

We define convex functions as in [18] so that they can take values $\pm\infty$

in addition to the reals. We consider two approaches to the problem. To elaborate on the first approach, we let $E(k)$ denote the epigraph of $k$ in $H$:

$$E(k)=\{(s,\mu): s\in I,\ -\infty<\mu<\infty,\ \mu\geqslant k(s)\}.$$

We define $k$ in $H$ to be a convex function if $E(k)$ is convex as a subset of $R^2$ [18, p. 23]. Let $K_c$ denote this class of functions. Theorem 4.2 of [18] states that $k$ is convex if and only if $k(\lambda s+(1-\lambda)t)<\lambda\gamma+(1-\lambda)\,\delta$, whenever $k(s)<\gamma$, $k(t)<\delta$, where $\gamma,\delta$ are reals and $0\leqslant\lambda\leqslant 1$. We now present the following simpler condition for convexity.

LEMMA 3.1. *$k$ is convex if and only if*

$$k(\lambda s+(1-\lambda)t)\leqslant\lambda k(s)+(1-\lambda)\,k(t), \tag{3.2}$$

*whenever $k(s)<\infty$, $k(t)<\infty$, and $0\leqslant\lambda\leqslant 1$.*

*Proof.* Assume $k$ is convex, i.e., $E(k)$ is a convex set. Let $k(s)<\infty$ and $k(t)<\infty$. Let $(\gamma_n)$ and $(\delta_n)$ be sequences of real numbers such that $k(s)<\gamma_n$, $k(t)<\delta_n$, $\gamma_n\to k(s)$, and $\delta_n\to k(t)$. Then $(s,\gamma_n)$ and $(t,\delta_n)$ are in $E(k)$ and hence, by convexity of $E(k)$, we conclude that $(\lambda s+(1-\lambda)t,\ \lambda\gamma_n+(1-\lambda)\,\delta_n)$ is in $E_k$. Thus, $k(\lambda s+(1-\lambda)t)\leqslant\lambda\gamma_n+(1-\lambda)\,\delta_n$ for all $n$, and by taking limits (3.2) follows. Conversely, assume (3.2) holds. If $(s,\gamma)$, $(t,\delta)$ are in $E_k$, then $k(s)\leqslant\gamma<\infty$ and $k(t)\leqslant\delta<\infty$. Thus, $\lambda k(s)+(1-\lambda)\,k(t)\leqslant\lambda\gamma+(1-\lambda)\,\delta<\infty$ and, by (3.2), $(\lambda s+(1-\lambda)t,\ \lambda\gamma+(1-\lambda)\,\delta)$ is in $E_k$. Thus $E_k$ is a convex set and $k$ is convex. The proof is complete.

LEMMA 3.2. *$K_c\subset K_q$ and $K_c=\bar{K}_c$.*

*Proof.* Let $k\in K_c$, $s$, $t\in I$, and $0\leqslant\lambda\leqslant 1$. If $k(s)=\infty$ or $k(t)=\infty$ then (3.1) holds. Otherwise, $\lambda k(s)+(1-\lambda)\,k(t)\leqslant\max\{k(s),k(t)\}$ and (3.1) follows from (3.2). Thus, $K_c\subset K_q$. The assertion $K_c=\bar{K}_c$ may be established by taking a convergent sequence in $K_c$. The proof is complete.

The above lemma shows that condition (2) holds for any $P\subset K_c$ since it holds for $K_q$. Also conditions (1) and (2) hold for $P=K_c$ for all $1\leqslant p\leqslant\infty$. The following lemma enables us to develop the second approach to our problem by providing an alternative definition of convex functions.

LEMMA 3.3. *Let $k\in K_c\cap L_p$, $1\leqslant p\leqslant\infty$; then $k>-\infty$. Hence, $k$ in $L_p$ is convex if and only if $k>-\infty$ and*

$$k(\lambda s+(1-\lambda)t)\leqslant\lambda k(s)+(1-\lambda)\,k(t) \tag{3.3}$$

*for all $s$, $t$ in $I$ and $0\leqslant\lambda\leqslant 1$. Furthermore $k<\infty$ on $(a,b)$.*

*Proof.* Suppose that $k \in K_c \cap L_p$ and $k(s) = -\infty$ for some $s$ in $I$. If $|k(t)| = \infty$ for all $t$ in $I$ then $k \notin L_p$. Hence assume that $|k(t)| < \infty$ for some $t$ in $I$. Without loss of generality assume that $s < t$. By (3.2) we conclude that $k = -\infty$ on $[s, t)$ and $k \notin L_p$. Hence $k > -\infty$. Now (3.3) is identical to (3.2) when $k(s) < \infty$ and $k(t) < \infty$. When $k(s) = \infty$ or $k(t) = \infty$, then (3.3) clearly holds since $k > -\infty$. Alternatively, equivalence of (3.3) follows from Theorem 4.1 of [18].

By convexity of $E(k)$ or (3.3), if $k(s) = \infty$ for some $s$ in $(a, b)$ then $k = \infty$ on $[a, s]$ or $[s, b]$ and $k \notin L_p$. Thus $k < \infty$ on $(a, b)$. This also follows from the fact that $K_c \subset K_q$. The proof is complete.

Inequality (3.3) corresponds to the usual definition of a real-valued convex function. Because $k > -\infty$, it avoids terms such as $\infty - \infty$ when $k = \infty$. The second approach to our problem involving convex functions is to define $k$ to be convex if $k > -\infty$ and it satisfies (3.3). Thus, let $K'_c$ be the set of all so defined convex functions $k$ in $H$. We may then show that $K'_c \subset K_q$ and $K'_c \cap L_p = \bar{K}'_c \cap L_p$ for all $1 \leqslant p \leqslant \infty$. The former is obvious, the proof of the latter is similar to that of Lemma 3.3. Thus, condition (2) holds for any $P \subset K'_c$, and conditions (1) and (2) hold for $P = K'_c$ for all $1 \leqslant p \leqslant \infty$. We remark that certain properties of sequences of convex functions are established in [28].

(iii) *Super-Additive Functions*

Let $I = [a, b]$ with $a < 0 < b$ and $a + b \geqslant 0$. A function $k$ in $H$ is said to be super-additive if

$$k(s+t) \geqslant k(s) + k(t) \tag{3.4}$$

holds whenever $s, t, s+t$ are in $I$ and $k(s) > -\infty$, $k(t) > -\infty$. (This means that when, for example, $k(s) = +\infty$ and $k(t) = -\infty$ and, consequently, the value of $k(s) + k(t)$ is left undefined, the value of $k(s+t)$ is not restricted by (3.4). See [10].) Let $K_a$ be the set of all super-additive functions on $I$. Generally these functions are defined on unbounded intervals $(0, \infty)$, $(-\infty, 0)$, or $(-\infty, \infty)$; however, for our purpose we define them on the compact interval $I$. If $k$ is super-additive and $h(s) = k(s - c)$, where $c$ is real, then $h$ is not necessarily super-additive on $[a + c, b + c]$. Hence, the properties of these functions depend upon their domain of definition. It is known that there are nonmeasurable functional solutions to (3.4). However, since we are interested in functions in $L_p$, we consider only measurable super-additive functions.

PROPOSITION 3.2. *Let $P \subset K_a$ be such that $k \geqslant h$ a.e. for all $k$ in $P$, where $h$ is an a.e. finite measurable function on $I$. Also suppose that $k(s) \geqslant -Cs$ for all $0 < s < \varepsilon$ uniformly for all $k$ in $P$ where $C > 0$ and $0 < \varepsilon < b$. Then condi-*

*tion* (3) *holds for* $P$ *with* $r=1$, *i.e., with* $a=x_0<x_1=b$ *for all* $1\leqslant p<\infty$. *If* $p=\infty$ *then condition* (3) *holds for any* $P\subset K_a$ *provided that* $k(s)\geqslant -Cs$ *for all* $0<s<\varepsilon$ *uniformly for all* $k$ *in* $P$.

*Proof.* Let $1\leqslant p<\infty$ and let $\|k\|_p\leqslant D$ for all $k\in B\subset P\cap L_p$. We show that functions in $B$ are uniformly bounded on $[c, d]\subset(a, b)$. This part of the proof uses certain ideas from [10], suitably modified to apply to our setting in $L_p$ and augmented by additional arguments. We first make one observation. Suppose that $0<u<b$ and $k(u)=2\rho$ where $-\infty\leqslant\rho<\infty$. If $s+t=u$, $s>0$, $t>0$ then $k(u)\geqslant k(s)+k(t)$. Hence $k(s)\leqslant\rho$ or $k(t)\leqslant\rho$. Consequently, if $E=\{s\in(0, u)\colon k(s)\leqslant\rho\}$, then $(0, u)=E\cup(u-E)$. It follows that $\mu(E)\geqslant u/2$. Now assume that $c>0$. We show that functions in $B$ are uniformly bounded below on $[c, d]$. Otherwise, there exist sequences $(k_n)$ in $B$ and $(u_n)$ in $[c, d]$ such that $u_n\to u$ in $[c, d]$ and $k_n(u_n)\leqslant -2n$. By the above observation, if $E_n=\{s\in(0, d)\colon k_n(s)\leqslant -n\}$, then $\mu(E_n)\geqslant c/2>0$ for all $n$. Let $G_n=\{s\in(0, d)\colon h(s)\leqslant k_n(s)\}$ and $G=\bigcap_n G_n$. Then, by hypothesis, $\mu(G)=d$. If $F_n=\{s\in(0, d)\colon h(s)\leqslant -n\}$, then $F_n\cap G\supset E_n\cap G$. Hence $\mu(F_n)\geqslant\mu(E_n)\geqslant c/2$. But $F_n\supset F_{n+1}$, and hence $\mu(F)\geqslant c/2$ where $F=\bigcap_n F_n=\{s\in(0, d)\colon h(s)=-\infty\}$. This is a contradiction since $h$ is finite a.e. Thus $B$ is uniformly bounded below on $[c, d]$, where $c>0$. The proof for a uniform lower bound on $[c, d]$ when $d<0$ is similar. Now assume $c<0<d$. Choose $\delta$ so that $c<-2\delta<0<2\delta<d$. Then if $u\in[-\delta, \delta]$, there exist $s$ in $[c, -\delta]$, $t$ in $[\delta, d]$ so that $u=s+t$ and $k_n(u)\geqslant k_n(s)+k_n(t)$. Since $(k_n)$ is uniformly bounded below on $[c, -\delta]$ and $[\delta, d]$, so it is on $[-\delta, \delta]$ and hence on $[c, d]$. Thus the uniform lower bound in all cases is established. (In fact, $(k_n)$ is uniformly bounded below on $I$.)

To show the upper bound assume that $B$ is not uniformly bounded above on $[c, d]$. Then there exist sequences $(k_n)$ in $B$ and $(v_n)$ in $[c, d]$ such that $v_n\to v$ in $[c, d]$ and $k_n(v_n)\geqslant n$. Let $0<\sigma<\min\{b-v, v-a\}/5$. Choose $N>0$ so that $v-\sigma\leqslant v_n\leqslant v+\sigma$ for all $n\geqslant N$. Let $s$ satisfy $v+2\sigma\leqslant s\leqslant v+3\sigma<b$. If $t_n=s-v_n$, then, clearly, $\sigma\leqslant t_n\leqslant 4\sigma<b$. Since $s=v_n+t_n$, we have $k_n(s)\geqslant k_n(v_n)+k_n(t_n)$. Now, by the first part, $k_n(t)\geqslant M>-\infty$ for all $n$ for all $t$ in $[\sigma, 4\sigma]$. Hence, $k_n(s)\geqslant n+M$ for all $s$ in $J=[v+2\sigma, v+3\sigma]$. If $\chi$ is the indicator function of $J$, then we have

$$\|k_n\|_p\geqslant\|k_n\chi\|_p\geqslant\max\{n+M, 0\}\,\sigma^{1/p}.$$

It follows that $\|k_n\|_p$ are not bounded, which is a contradiction. Thus a uniform upper bound is established.

We now show that the total variation of $k$ on $[c, d]$ is uniformly bounded for all $k$ in $B$. Assume, without loss of generality, that $c\leqslant 0\leqslant d$ and $c+d\geqslant 0$. First consider the interval $[0, d]$. Indeed, let $0=s_0<s_1<\cdots<s_n=d$ be any partition of $[0, d]$. By combining adjacent intervals, if necessary, first assume that $k(s_{i-1})\leqslant k(s_i)\geqslant k(s_{i+1})$, $i=1, 3, ..., n-2$, and

$k(s_{n-1}) \leqslant k(s_n)$, where $n$ is odd. Let $\lambda_i = s_{i+1} - s_i > 0$. Now, if $0 \leqslant s < t < b$, then $t - s < b$ and we have by super-additivity, $k(t) - k(s) \geqslant k(t-s)$. Thus, $0 \geqslant k(s_{i+1}) - k(s_i) \geqslant k(\lambda_i)$ for $i = 1, 3, ..., n-2$. Hence,

$$\sum |k(s_{i+1}) - k(s_i)| \leqslant \left|\sum k(\lambda_i)\right| \leqslant C\lambda,$$

where $\lambda = \sum \lambda_i$ and all summations are over indexes $i = 1, 3, ..., n-2$. (If $0 < t - s < \varepsilon$ then $k(t) - k(s) \geqslant k(t-s) \geqslant -C(t-s)$ for all $k$ in $P$ by the second assumption on $P$. This implies that $k(s) \geqslant -Cs$ for all $0 < s \leqslant b$ for all $k$ in $P$ and, in particular, $k(\lambda_i) \geqslant -C\lambda_i$ for all $i$. Thus functions in $P$ are uniformly bounded below on $(0, b]$ by $-Cb$. Hence, the lower bounding function $h$ is really effective on $[a, 0]$.) Now, if $0 \leqslant s < t < u < b$, then as above $k(u) - k(t) \geqslant k(u-t)$ and hence $k(t) - k(s) \leqslant k(u) - k(s) - k(u-t)$. We therefore have

$$0 \leqslant k(s_i) - k(s_{i-1}) \leqslant k(s_{i+1}) - k(s_{i-1}) - k(\lambda_i),$$

for $i = 1, 3, ..., n-2$. Summing the above and combining with the previous inequality, we have,

$$\sum |k(s_{i+1}) - k(s_i)| \leqslant |k(s_{n-1}) - k(s_0)| + 2C\lambda + |k(s_n) - k(s_{n-1})|,$$

where the summation is over all indexes $1, 2, ..., n-1$. Note that $0 < \lambda \leqslant d$ and, hence, if $M$ is the uniform bound on $|k|$ on $[c, d]$, then $\sum |k(s_{i+1}) - k(s_i)| \leqslant 4M + 2Cd$. All other cases, for example, having $k(s_{i-1}) \geqslant k(s_i) \leqslant k(s_{i+1})$, $i = 1, 3, ..., n-1$, where $n$ is even, or these inequalities for $i = 1, 3, ..., n-2$ and $k(s_{n-1}) \geqslant k(s_n)$, where $n$ is odd, may be considered similarly. Thus, we have shown that all $k$ in $B$ have uniformly bounded total variation on $[0, d]$. A similar conclusion may be drawn for $[c, 0]$ and therefore for $[c, d]$. (Since $c + d \geqslant 0$, the proof for $[c, 0]$ is identical to that for $[0, d]$ as $\lambda_i \leqslant -c$ implies $\lambda_i \leqslant d$.) Now consider the case $p = \infty$. If $k \in B \subset P \cap L_\infty$, then $k \geqslant -D$ a.e. for all $k$ in $B$. With this observation, the results for this case may be proved as above or otherwise. (Note that in this case one may show that $-2D \leqslant k \leqslant D$ on $(a, b)$ and $k(b) \geqslant -2D$.) The proof is complete.

We note that if $k \in K_a \cap L_p$, $1 \leqslant p \leqslant \infty$, then $|k| < \infty$ on $(a, b)$ by a proof as in Proposition 3.2. For such a $k$ we have $0 \geqslant k(0) \geqslant k(s) + k(-s)$ for $0 < s \leqslant -a \leqslant b$. Hence the assumption that $k(s) \geqslant -Cs$ for $0 < s < \varepsilon$ as in Proposition 3.2 implies that $k(s) \leqslant -Cs$ for $a \leqslant s \leqslant 0$ in addition to $k(s) \geqslant -Cs$ for $0 < s \leqslant b$. Thus the functions in $P$ are uniformly bounded above on $[a, 0]$ by $-Ca$. Let $L_a \subset K_a$ be the set of all $k$ in $K_a$ such that $k \geqslant h$ a.e. and $k(s) \geqslant -Cs$ for $0 < s < \varepsilon$ where $h$, $C$ and $\varepsilon$ are as in the statement of Proposition 3.2. Again, let $M_a \subset K_a$ be the set of all $k$ in $K_a$ such that $k(s) \geqslant -Cs$ for $0 < s < \varepsilon$. Then, clearly, $\bar{L}_a = L_a$ and $\bar{M}_a = M_a$. Hence, condition (1) holds for $L_a$ for $1 \leqslant p < \infty$, and for $M_a$ if $p = \infty$.

(iv) *Star-Shaped Functions*

In this case we consider $I = [0, b]$. A function $k$ in $H$ is star-shaped if $k(\lambda s) \leqslant \lambda k(s)$ for all $s$ in $I$, and all $0 \leqslant \lambda \leqslant 1$ [3]. Equivalently, $k$ is star-shaped if $k(0) \leqslant 0$ and $k(s)/s \leqslant k(t)/t$ whenever $0 < s \leqslant t \leqslant b$. Let $K_s$ be the set of all star-shaped functions.

PROPOSITION 3.3. *For any $P \subset K_s$ condition* (3) *holds with $a = 0$ and $r = 1$, i.e., with $0 = x_0 < x_1 = b$, for all $1 \leqslant p \leqslant \infty$.*

*Proof.* Assume first that $1 \leqslant p < \infty$. Let $\|k\|_p \leqslant D$ for all $k \in B \subset P \cap L_p$. We show that functions in $B$ are uniformly bounded on $[c, d] \subset (0, b)$. Let $h(s) = k(s)/s$, $0 < s \leqslant b$. Then $h$ is nondecreasing on $(0, b]$. Let $\chi$ be the indicator function of $[c, d]$. Since $|k(s)| \geqslant c\,|h(s)|$ for all $s$ in $[c, d]$ and $h$ is nondecreasing on $(0, b]$, we have for all $k$ in $B$,

$$\begin{aligned} D &\geqslant \|k\|_p \geqslant \|k\chi\|_p \geqslant c\,\|h\chi\|_p \\ &\geqslant c \max\{|h(c)|, |h(d)|\} \min\{c^{1/p}, (b-d)^{1/p}\}. \end{aligned}$$

It follows that $\max\{|h(c)|, |h(d)|\} \leqslant A$ for some $A$ which is independent of $k$. By monotonicity of $h$, we have $|h(s)| \leqslant \max\{|h(c)|, |h(d)|\}$ for all $s$ in $[c, d]$, and this gives $|k(s)| \leqslant d\,|h(s)| \leqslant dA$ for all $s$ in $[c, d]$ for all $k$ in $B$. Thus a uniform bound is established.

Now we assert that the total variation of $k$ on $[c, d]$ is uniformly bounded for all $k$ in $B$. This follows at once from the fact that $k(s) = sh(s)$ on $[c, d]$ where $h$ is nondecreasing and hence of total variation $h(d) - h(c)$ on $[c, d]$. Using an elementary argument or Theorem 3 of [15, p. 216], we conclude that the total variation on $[c, d]$ of any $k$ in $B$ is bounded by

$$(d-c) \sup\{|h(t)| : t \in [c, d]\} + d\,|h(d) - h(c)| \leqslant (3d - c) A,$$

where $A$, as shown above, is independent of $k$. The case for $p = \infty$ is simpler and may be similarly proved.

It is easy to see that $K_s = \bar{K}_s$ and thus condition (1) holds for $P = K_s$ for all $1 \leqslant p \leqslant \infty$ in addition to condition (3) as shown above. If $k \in K_s \cap L_p$, any $1 \leqslant p \leqslant \infty$, then $k > -\infty$ on $(0, b]$ and $k < \infty$ on $[0, b)$.

We now state three results from [10] which present a comparison of some function classes.

(a) If $k$ is star-shaped on $(0, b)$, i.e., $k(s)/s$ is nondecreasing, then $k$ is super-additive, but need not be convex or concave on $(0, b)$.

(b) If $k$ is concave and super-additive on $(0, b)$ then $k$ is star-shaped.

(c) A necessary and sufficient condition that a convex function $k$ be super-additive on $(0, b)$ is that $k(0+) \leqslant 0$.

(v) *Monotone (Nondecreasing) Functions*

$k$ in $H$ is monotone if $k(s) \leqslant k(t)$ for all $s \leqslant t$. Let $K_{\mathrm{m}}$ denote this class. Then, clearly condition (2) holds for any $P \subset K_{\mathrm{m}}$. Also $K_{\mathrm{m}} = \bar{K}_{\mathrm{m}}$; thus conditions (1) and (2) hold for $P = K_{\mathrm{m}}$ for all $1 \leqslant p \leqslant \infty$. Note that $K_{\mathrm{m}} \subset K_{\mathrm{q}}$. If $k \in K_{\mathrm{m}} \cap L_p$, any $1 \leqslant p \leqslant \infty$, then $k > -\infty$ on $(a, b]$ and $k < \infty$ on $[a, b)$. This is the simplest case of functions under consideration.

(vi) *n-Convex Functions*

A real-valued function $k$ on $(a, b)$ is called an $n$-convex function $(n \geqslant 1)$ if for all choices of $n$ or $n+1$ points $\{s_i\}$ any of the following three equivalent conditions holds.

(a) If $a < s_1 < s_2 < \cdots < s_n < b$ then $(-1)^{n+i+1}\,(P(s) - k(s)) \geqslant 0$ for all $s$ in $(s_i, s_{i+1})$, $1 \leqslant i \leqslant n-1$, where $P(s)$ is the unique Lagrange interpolating polynomial of degree at most $(n-1)$ passing through the points $(s_i, k(s_i))$, $1 \leqslant i \leqslant n$.

(b) If $a < s_0 < s_1 < \cdots < s_n < b$, then

$$\det \begin{bmatrix} 1 & s_0 & \cdots & s_0^{n-1} & k(s_0) \\ 1 & s_1 & \cdots & s_1^{n-1} & k(s_1) \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & s_n & \cdots & s_n^{n-1} & k(s_n) \end{bmatrix} \geqslant 0.$$

(c) If $a < s_0 < s_1 < \cdots < s_n < b$, then the $n$th order divided difference $[s_0, s_1, ..., s_n; k]$ of $k$ is nonnegative. (For a definition of the divided difference see [5, p. 40] or [17, p. 237]).

Equivalence of these definitions may be established by elementary methods or comparing definitions in [17]. We observe that 1-convex and 2-convex functions are, respectively, real-valued nondecreasing and convex functions on $(a, b)$. More complex cases of $n$-convex functions occur for $n \geqslant 3$. Since about 1940 there has been a considerable literature on $n$-convex functions and, more generally, functions which are convex with respect to an $n$-parameter family or a $\lambda_n$-family of functions. Some basic references are listed in Section 1; a brief survey appears in [17]. As it is not our purpose to delve deeply into this area in this work, we merely state two known properties of $n$-convex functions which can be derived directly from definition (a).

(1) If $\{s_i\}$ and $P(s)$ are as in definition (a), then $(P(s)-k(s))\leqslant 0$ on $(s_n, b)$ and $(-1)^{n-1}(P(s)-k(s))\geqslant 0$ on $(a, s_1)$.

(2) For every $n$-convex function $k$ there exist an integer $r$, $1\leqslant r\leqslant n$, and points $\{x_i\}$ with $a=x_0<x_1<\cdots<x_r=b$ such that the following holds: If $r=n$, then $(-1)^{n+i}k$ is nondecreasing on $(x_{i-1}, x_i)$ for all $1\leqslant i\leqslant n$. If $r<n$, then $(-1)^{r+i}k$ (or equivalently $(-1)^i k$) is nondecreasing on $(x_{i-1}, x_i)$ for all $i$ or nonincreasing on $(x_{i-1}, x_i)$ for all $i$. The integer $r$ and points $\{x_i\}$ depend upon $k$.

We let $K_n$ denote all functions $k$ in $H$ such that $k$ is (real-valued) $n$-convex on $(a, b)$. It follows at once from property (2) above that condition (2) holds for any $P\subset K_n$ with $1\leqslant r\leqslant z=n$. The following proposition shows that condition (1) holds for $P=K_n$ for all $1\leqslant p\leqslant\infty$.

PROPOSITION 3.4. $K_n\cap L_p=\bar{K}_n\cap L_p$, $1\leqslant p\leqslant\infty$.

*Proof.* Let $k\in\bar{K}_n\cap L_p$, we show that $k\in K_n$. We assert that $|k|<\infty$ on $(a, b)$. Suppose that $k(t)=-\infty$ for some $t$ in $(a, b)$; then we reach a contradiction as shown below. There exists a point, say, $t_n$ in $(t, b)$ such that $|k(t_n)|<\infty$, otherwise $k\notin L_p$, for any $p$. Set $t_{n-1}=t$. Again, there exists a point $t_{n-2}$ in $(a, t_{n-1})$ such that $|k(t_{n-2})|<\infty$, otherwise $k\notin L_p$. Arguing in this manner, we have $a=t_0<t_1<\cdots<t_n<t_{n+1}=b$ such that $|k(t_i)|<\infty$, $i=1, 2, ..., n-2, n$, and $k(t_{n-1})=-\infty$. Now, since $k\in\bar{K}_n$, there exists a sequence $(k_m)$ in $K_n$ such that $k_m\to k$ pointwise on $I$. Let $P_m(s)$ be the interpolating Lagrange polynomial passing through $(t_i, k_m(t_i))$, $1\leqslant i\leqslant n$, as in definition (a). We state the formula for $P_m(s)$ [5, p. 33]. Let

$$L_i(s)=\frac{(s-t_1)(s-t_2)\cdots(s-t_{i-1})(s-t_{i+1})\cdots(s-t_n)}{(t_i-t_1)(t_i-t_2)\cdots(t_i-t_{i-1})(t_i-t_{i+1})\cdots(t_i-t_n)}.$$

Then,

$$P_m(s)=\sum k_m(t_i)\,L_i(s)\ \left(\sum\text{ over }1\leqslant i\leqslant n\right).$$

We have $k_m(t_i)\to k(t_i)$ for all $i$. Clearly, $L_{n-1}(s)<0$ for all $s$ in $(t_n, b)$. Since $k_m(t_{n-1})\to-\infty$, we conclude that $P_m(s)\to\infty$ for all $s$ in $(t_n, b)$. Now, by property (1) we have $k_m(s)\geqslant P_m(s)$ on $(t_n, b)$ for all $m$. It follows that $k(s)=\infty$ on $(t_n, b)$. Thus $k\notin L_p$, a contradiction and hence $k>-\infty$ on $(a, b)$. In a similar manner, by assuming that $k(t)=\infty$ for some $t$, we may show that $k<\infty$ on $(a, b)$. In this case, we determine $\{t_i\}$ as above with $a=t_0<t_1<\cdots<t_n<t_{n+1}=b$, $t_n=t$, $|k(t_i)|<\infty$, $1\leqslant i\leqslant n-1$, and $k(t_n)=\infty$.

Now let $\{s_i\}$ be any $n$ points in $(a, b)$ as in definition (a). If $P'$, $P'_m$ are the interpolating Lagrange polynomials passing through $(s_i, k(s_i))$ and $(s_i, k_m(s_i))$, respectively, then since $k_m \to k$ on $I$, we have that $P'_m \to P'$ on $(a, b)$. Since definition (a) holds for $P'_m, k_m$ for all $m$, and $k$ is finite on $(a, b)$, in the limit it must also hold for $P', k$. Thus $k$ is $n$-convex on $(a, b)$ and it follows that $k \in K_n$. The proof is complete.

We summarize by applying Theorems 2.1 and 2.2 to the above classes of functions.

THEOREM 3.1. *Let $P \subset K_x$, where $x =$ q, c, s, m and n. Assume $P$ satisfies condition* (1) *for some $1 \leqslant p \leqslant \infty$. Then the following* (a) *and* (b) *apply to $P$ for $p$.*

(a) *If $(k_n)$ is a sequence of functions in $P \cap L_p$ such that $\|k_n\|_p \leqslant D$ for all $n$ and for some $D > 0$, then there exists a subsequence $(g_j)$ of $(k_n)$ and a $g$ in $P \cap L_p$ such that $(g_j)$ converges pointwise to $g$ on $I$ and $\|g\|_p \leqslant D$.*

(b) *$P \cap L_p$ is closed in $L_p$ and a best approximation to $f$ in $L_p$ from $P \cap L_p$ exists.*

In particular, since $P = K_x$, $x =$ q, c, s, m, n, satisfies condition (1) for all $1 \leqslant p \leqslant \infty$, the above conclusions (a) and (b) are applicable to $P = K_x$ for all $1 \leqslant p \leqslant \infty$.

The results for $K_a$ are somewhat different. Let $h$ be an a.e. finite measurable function on $I$ and $0 < \varepsilon < b$. Let $P \subset K_a$ be such that $k \geqslant h$ a.e. for all $k$ in $P$ and $k(s) \geqslant -Cs$ for all $0 < s < \varepsilon$ some $C > 0$ for all $k$ in $P$. Assume $P$ satisfies condition (1) for some $1 \leqslant p < \infty$. Then conclusions (a) and (b) apply to $P$ for $p$. If $p = \infty$ and condition (1) holds for $P \subset K_a$ where $k(s) \geqslant -Cs$ for all $0 < s < \varepsilon$ for all $k$ in $P$, then (a) and (b) with $p = \infty$ apply to $P$. Let $L_a \subset K_a$ be the set of all $k$ in $K_a$ such that $k \geqslant h$ a.e. and $k(s) \geqslant -Cs$ for $0 < s < \varepsilon$. Also let $M_a \subset K_a$ be the set of all $k$ in $K_a$ such that $k(s) \geqslant -Cs$ for $0 < s < \varepsilon$. Then $L_a$ and $M_a$ satisfy condition (1) for all $1 \leqslant p < \infty$ and $p = \infty$ respectively, and hence, conclusions (a) and (b) apply to $P = L_a$ for all $1 \leqslant p < \infty$ and to $P = M_a$ for $p = \infty$.

## ACKNOWLEDGMENT

The author is grateful to Professor R. M. Mathsen for discussions on $n$-convex and related classes of functions.

## REFERENCES

1. T. ANDO AND I. AMENIA, Almost everywhere convergence of prediction sequences in $L_p$ $(1 < p < \infty)$, *Z. Wahrsch. Verw. Gebiete* **4** (1965), 113–120.

2. E. F. Beckenbach and R. H. Bing, On generalized convex functions, *Trans. Amer. Math. Soc.* **58** (1945), 220–230.
3. A. M. Bruckner and E. Ostrow, Some function classes related to the class of convex functions, *Pacific J. Math.* **12** (1962), 1203–1215.
4. P. S. Bullen, A criterion for $n$-convexity, *Pacific J. Math.* **36** (1971), 81–98.
5. P. J. Davis, "Interpolation and Approximation," Dover, New York, 1975.
6. E. Deak, Über konvexe und interne Funktionen, sowie eine gemeinsame Verallgemeinerung von beiden, *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.* **5** (1962), 109–154.
7. F. Deutsch, Existence of best approximations, *J. Approx. Theory* **28** (1980), 132–154.
8. N. Dunford and J. T. Schwartz, "Linear Operators, Part I," Interscience, New York, 1958.
9. P. Hartman, Unrestricted $n$-parameter families, *Rend. Circ. Mat. Palermo (2)* **7** (1958), 123–142.
10. E. Hille and R. S. Phillips, "Functional Analysis and Semi-groups," Amer. Math. Soc., Providence, RI, 1957.
11. J. H. B. Kemperman, On the regularity of generalized convex functions, *Trans. Amer. Math. Soc.* **135** (1969), 69–93.
12. D. Landers and L. Rogge, Isotonic approximation in $L_s$, *J. Approx. Theory* **31** (1981), 199–223.
13. D. Landers and L. Rogge, Natural choice of $L_1$-approximants, *J. Approx. Theory* **33** (1981), 268–280.
14. R. M. Mathsen, $\lambda(n)$-convex functions, *Rocky Mountain J. Math.* **2** (1972), 31–43.
15. I. P. Natanson, "Theory of Functions of a Real Variable," Vol. 1, Ungar, New York, 1964.
16. J. Ponstein, Seven kinds of convexity, *SIAM Rev.* **9** (1967), 115–119.
17. A. W. Roberts and D. E. Varberg, "Convex Functions," Academic Press, New York, 1973.
18. R. T. Rockafellar, "Convex Analysis," Princeton Univ. Press, Princeton, NJ, 1970.
19. R. A. Rosenbaum, Sub-additive functions, *Duke Math. J.* **17** (1950), 227–247.
20. T. Shintani and T. Ando, Best approximants in $L^1$ space, *Z. Wahrsch. Verw. Gebiete* **33** (1975), 33–39.
21. P. W. Smith and J. J. Swetits, Best approximation by monotone functions, *J. Approx. Theory* **49** (1987), 398–403.
22. L. Tornheim, On $n$-parameter families of functions and associated convex functions, *Trans. Amer. Math. Soc.* **69** (1950), 457–467.
23. V. A. Ubhaya, An $O(n)$ algorithm for discrete $n$-point convex approximation with applications to continuous case, *J. Math. Anal. Appl.* **72** (1979), 338–354.
24. V. A. Ubhaya, Generalized isotone optimization with applications to starshaped functions, *J. Optim. Theory Appl.* **29** (1979), 559–571.
25. V. A. Ubhaya, Lipschitz condition in minimum norm problems on bounded functions, *J. Approx. Theory* **45** (1985), 201–218.
26. V. A. Ubhaya, Quasi-convex optimization, *J. Math. Anal. Appl.* **116** (1986), 439–449.
27. V. A. Ubhaya, Uniform approximation by quasi-convex and convex functions, *J. Approx. Theory* **55** (1988), 326–336.
28. V. A. Ubhaya, $L_p$ Approximation by quasi-convex and convex functions, *J. Math. Anal. Appl.* **139** (1989), 574–585.
29. V. A. Ubhaya, An $O(n)$ algorithm for least squares quasi-convex approximation, *Comput. Math. Appl.* **14** (1987), 583–590.
30. D. Zwick, Existence of best $n$-convex approximations, *Proc. Amer. Math. Soc.* **97** (1986), 273–276.

*Printed in Belgium*